ENDTERM EXAM 2019: COMPUTER SCIENCE II (B.MATH. 2ND YEAR)

- *This is a pen-and-paper, closed book exam. Use of computers/calculators is not allowed.*
- *All questions are compulsory. The question paper is of* 120 *points. The maximum you can score is* 100 *(that is, the scores above* 100 *are counted as* 100*).*

Problem 1 ($2 \times 10 = 20$ marks)**.** Answer the following True or False questions. For a False answer, please justify with a counter example or an explanation. For a True answer, you can just mention it is True, no justification is required.

(1) `print tan(1)` just prints `tan(1)` in `SageMath` since `SageMath` works symbolically and not numerically unless specifically asked to.

(2) The following short program prints integers from 1 to 10:

```
for i in range(10):
    print i
```

(3) By default, `SageMath` rounds real numbers to nearest even digit when needed (e.g. `1.001` in binary is rounded to `1.00` when working with precision of 3 bits, while 1.011 will be rounded to 1.10 in the same situation): this is just a quirk in the implementation and does not serve any benefit.

(4) Given N data points (x_i, y_i) for $1 \leq i \leq N$ where $x_i \in \mathbb{R}^r$ is a r-tuple and $y_i \in \mathbb{R}$. Then we can find the best fit linear relation F for $y = F(x)$ using QR-decomposition of an appropriate matrix. There does not exist a best fit linear relation when $x_i \in \mathbb{C}^r, y_i \in \mathbb{C}$ since QR-decomposition exists only for real matrices.

(5) For finding roots of a general function $f : \mathbb{R} \to \mathbb{R}$ which is infinitely differentiable, Newton-Raphson method has a faster rate of convergence than the fixed point iteration.

(6) Consider a tridiagonal square matrix of size $N \times N$, that is a matrix $A = (a_{ij})_{1 \leq i,j \leq N}$ where $a_{ij} \neq 0 \Rightarrow |i - j| \leq 1$. Assume that $|a_{ii}| > |a_{i,i-1}| + |a_{i,i+1}|$ for all $1 \leq i \leq N$ where we define $a_{1,-1} = a_{N,N+1} := 0$. Then A^{-1} exists.

(7) It is better to use high degree polynomials for interpolation as the size of data increases as it gives a closer approximation for the intermediate values.

(8) A Gaussian quadrature which computes weighted sum of n evaluations has zero error when computing the integral for a polynomial of degree $\leq 2n - 1$.

(9) A local truncation error of order h, where h is the step-size, is enough to guarantee convergence and consistency for numerical solutions of ordinary differential equations.

(10) There is exactly one Runge-Kutta method which has two stages and whose local truncation error is of order 3.

Problem 2 (10 marks)**.** The following `SageMath` function was supposed to produce the derivative of the piecewise polynomial function $p(x) := \begin{cases} x^2 - x + 1 & x < 1 \\ 2x^2 - 3x + 2 & x \geq 1 \end{cases}$ which is differentiable everywhere. However the program produces incorrect output. Explain what went wrong and what to expect as the output instead.

```
def p(x):
    if (x<1):
        return x*x-x+1
    else:
        return 2*x*x-3*x+2

print p(x).derivative() #Supposed to print the derivative!
```

Problem 3 (15 marks)**.** Let $f(x) = \cos(x)$. Write down it's Taylor series expansion around $x = 0$. Till how many terms should we sum the Taylor series to calculate $f(1/2)$ to an accuracy of 53 bits?

Problem 4 (15 marks)**.** Let $f : [a, b] \to \mathbb{R}$ be a function which is infinitely differentiable in some neigbhorhood of $[a, b]$. Let $L_n(f)$ be the degree n Lagrange polynomial approximating f, that is interpolating $n + 1$ points $(a_k := a + k\left(\frac{b-a}{n}\right), f(a_k))$ for $0 \le k \le n$. Show that

$$|f(x) - L_n(f)(x)| \le \frac{(b-a)^{n+1}}{(n+1)!} \sup_{z\in[a,b]} |f^{(n+1)}(z)|.$$

Problem 5 (15 marks)**.** Determine a, b, c so that the function:

$$f(x) = \begin{cases} 3 + x - 9x^3 & x \in [0, 1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [1, 2] \end{cases}$$

is a cubic spline for data points $(0, f(0)), (1, f(1)), (2, f(2))$. If furthermore, the boundary conditions are *natural* compute d.

Problem 6 (15 marks)**.** Compute QR decomposition of the following matrix

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix}$$

that is find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

Problem 7 (15 marks)**.** Assume $f : [a, b] \to \mathbb{R}$ is a function which is infinitely differentiable in some neighborhood of $[a, b]$. We want to compute $\int_a^b f(x)dx$ using Newton-Cotes integration rules:

(1) Derive trapezoidal rule for integration by replacing a function by its piecewise linear approximation.
(2) Derive Simpson's $\frac{1}{3}$-rule for integration by replacing a function by its piecewise quadratic approximation.

Problem 8 (15 marks)**.** Calculate solutions of the ordinary differential equation for a function $y(t)$ of $t \in [0, 1]$:

$$\frac{dy}{dt} = \lambda y \text{ with initial value } y(0) = 1$$

using (a) Euler's method and (b) Trapezoidal method. Use the result from the calculations to conclude that Euler's method is not A-stable while Trapezoidal method is A-stable.

—THE END—